In conclusion the author expresses her deep gratitude to N. N. Lebedev and Ia. S. Ufliand for helpful suggestions and discussions.

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Translated by E. D.

UDC 539.3

ON THE USE OF RATIONAL APPROXIMATING FUNCTIONS

PMM Vol. 36, №6, 1972, pp. 1136-1138
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(Received December 13, 1971)

Boundary value problems of the linear theory of viscoelasticity are solved using rational functions to approximate a function of the Poisson ratio. The problem of interpolation is solved for the class of rational fractions and the error of the approximation is estimated.

1. Suppose that a sufficiently smooth function $\varphi(\omega)$ of the real variable ω is to be approximated on the interval $a \leqslant \omega \leqslant b$ using another prescribed function, to a specified accuracy. The problem embraces that of interpolation, the latter consisting of finding the interpolation function $f_N(\omega)$ belonging to some class F and assuming, at the interpolation nodes, i. e. at certain prescribed points

$$\omega_0, \omega_1, \omega_2, \ldots, \omega_N \tag{1.1}$$

of the segment $[a, b]$, the same values as the function $\varphi(\omega)$, i. e.

$$f_N(\omega_0) = \varphi_0, \quad f_N(\omega_1) = \varphi_1, \ldots, \quad f_N(\omega_N) = \varphi \tag{1.2}$$

where

$$\varphi_n \equiv \varphi(\omega_n), \qquad n = 0, 1, \ldots, N \tag{1.3}$$

Depending on the class F, the interpolation problem may have an infinite number of solutions, or none. If polynomials of degree not greater than N are used to represent the function f_N, then the interpolation problem has a unique solution. In this case the polynomials are called the interpolation polynomials. Sometimes the properties of the function $\varphi(\omega)$ are such that it is more convenient to write the functions $f_N(\omega)$ in the form of rational fractions

$$f_N(\omega) = \sum_{i=0}^{M} p_i \omega^i / P_N, \qquad P_N = \sum_{i=1}^{N} q_i \omega^i \tag{1.4}$$

where p_i and q_i are constants. Obviously, the approximation (1.4) is more general than that employing the polynomials.

2. Remembering that the method is to be applied to the problems of the theory of viscoelasticity [1] we assume that the polynomial P_N has real nonnegative distinct roots and, that the numbers a and b are also nonnegative. The rational fraction $f_N(\omega)$ can be written as a sum of a polynomial and a proper rational fraction. The problem of interpolation using polynomials is well known [2]. We shall therefore consider the proper rational fractions, i.e. we assume that $M \leqslant N$.

Under the assumptions made, every function $f_N(\omega)$ of the class F can be written in the form

$$f_N(\omega) = \sum_{i=0}^{N} A_i \frac{1}{1+\beta_i\omega} \tag{2.1}$$

where A_i and β_i are constants. As β_i are assumed given, we say that the poles of the function $f_N(\omega)$ are fixed. Let us find the interpolation function $f_N(\omega)$ with fixed poles. To do this, we construct $N+1$ equations

$$\varphi_n = \sum_{i=0}^{N} A_i \frac{1}{1+\beta_i\omega_n}, \qquad n = 0, 1, \ldots, N \tag{2.2}$$

defining $N+1$ unknowns A_i. As every expression $(1+\beta_i\omega_n) > 0$ $(i, n = 0, 1, \ldots, N)$, the determinant of the system (2.2) is positive and (2.2) has a unique solution

$$A_k = \sum_{n=0}^{N} A_{kn}\varphi_n$$

$$A_{kn} = \frac{\prod_{i=0}^{N}(1+\beta_k\omega_i) \prod_{i=0\,(i\neq k)}^{N}(1+\beta_i\omega_k)}{\prod_{i=0\,(i\neq k)}^{N}(\beta_i-\beta_k) \sum_{i=0\,(i\neq n)}^{N}(\omega_i-\omega_n)} \tag{2.3}$$

Inserting (2.3) into (2.1) we obtain the interpolation formula for the present case. The uniqueness of (2.1) can be proved by induction.

3. Let us compute the error of this approximation. We assume that the function $\varphi(\omega)$ has an $(N+1)$-th derivative on the segment $a \leqslant \omega \leqslant b$. We set

$$R_N(\omega) = \varphi(\omega) - f_N(\omega) \tag{3.1}$$

and introduce an auxilliary function

$$v(\omega) = R_N(\omega) - KX_N(\omega) \tag{3.2}$$

where

$$X_N(\omega) \equiv \frac{\prod_{i=0}^{N}(\omega-\omega_i)}{\prod_{i=0}^{N}(1+\beta_i\omega)}, \qquad K \equiv \frac{R_N(\omega_*)}{X_N(\omega_*)} \tag{3.3}$$

and ω_* is a fixed point on the segment $[a, b]$ different from all the nodes (1.1). Thus the function $v(\omega)$ has $N+2$ roots on the segment $[a, b]$ and it follows that its $(N+1)$-th derivative has at least one root. Therefore at some point ξ we have

$$w^{(N+1)}(\xi)=0, \qquad w(\omega)\equiv v(\omega)\prod_{i=0}^{N}(1+\beta_i\omega) \tag{3.4}$$

Let us multiply both sides of (3.2) by the denominator of the expression for X_N and differentiate both sides $N+1$ times. Taking into account (3.4) we have, at the point ξ

$$K=\frac{Q(\xi)}{(N+1)!}, \quad Q(\xi)\equiv\frac{d^{N+1}}{d\omega^{N+1}}\Big[\varphi(\omega)\prod_{i=0}^{N}(1+\beta_i\omega)\Big]_{\omega=\xi} \tag{3.5}$$

Equating (3.5) and (3.3) we obtain

$$R_N(\omega_*)=\frac{Q(\xi)}{(N+1)!}X_N(\omega_*)$$

The point ω_* was chosen arbitrarily, hence

$$R_N(\omega)=\frac{Q(\xi)}{(N+1)!}\frac{\prod_{i=0}^{N}(\omega-\omega_i)}{\prod_{i=0}^{N}(1+\beta_i\omega)} \tag{3.6}$$

$$|R_N(\omega)|\leqslant\frac{M_N}{(N+1)!}\left|\frac{\prod_{i=0}^{N}(\omega-\omega_i)}{\prod_{i=0}^{N}(1+\beta_i\omega)}\right|$$

$$M_N=\max_{a\leqslant\omega\leqslant b}\left|\frac{d^{N+1}}{d\omega^{N+1}}\Big[\varphi(\omega)\prod_{i=0}^{N}(1+\beta_i\omega)\Big]\right| \tag{3.7}$$

4. Several remarks follow.

1. If in (1.4) $M=N$, we must set $\beta_0=0$.

2. If the term A_0/ω is included in (2.1), we must set $A_0=\beta_0 A_0'$ and perform the passage to the limit with $\beta_0\to\infty$.

3. If one of the poles is not assumed fixed, it can be found by increasing the number of nodes (1.1) by one. Then the system (2.2) must be solved from $N+2$ equations for $N+1$ unknowns A_i, and one unknown β_k. Let us denote

$$Q_n\equiv\prod_{i>j=0}^{n}(\omega_i-\omega_j), \qquad Q_{n,k}\equiv\frac{Q_n}{\prod\limits_{i>j=0\ (i=k\vee j=k)}^{n}(\omega_i-\omega_j)}$$

Then the formulas (2.3) require the following additional expression:

$$\beta_j=-\frac{\sum_{k=0}^{N+1}\Big\{\varphi_k Q_{N+1,k}\prod_{i=0\,(i\neq j)}^{N}(1+\beta_i\omega_k)\Big\}}{\sum_{k=0}^{N+1}\Big\{\varphi_k Q_{N+1,k}\omega_j\prod_{i=0\,(i\neq j)}^{N}(1+\beta_i\omega_k)\Big\}} \qquad (j=0,1,\ldots,N)$$

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Translated by L. K.